

A HOPF ALGEBRA ASSOCIATED TO A LIE PAIR

ZHUO CHEN, MATHIEU STIÉNON, AND PING XU

In tribute to Alan Weinstein on the occasion of his seventieth birthday

ABSTRACT. The quotient $L/A[-1]$ of a pair $A \hookrightarrow L$ of Lie algebroids is a Lie algebra object in the derived category $D^b(\mathcal{A})$ of the category \mathcal{A} of left $\mathcal{U}(A)$ -modules, the Atiyah class $\alpha_{L/A}$ being its Lie bracket. In this note, we describe the universal enveloping algebra of the Lie algebra object $L/A[-1]$ and we prove that it is a Hopf algebra object in $D^b(\mathcal{A})$.

1. INTRODUCTION

Let A be a Lie algebroid over a manifold M . Its space of smooth sections $\Gamma(A)$ is a Lie-Rinehart algebra over the commutative ring $R = C^\infty(M)$. By an A -module, we mean a module over the Lie-Rinehart algebra corresponding to the Lie algebroid A , i.e. a module over the associative algebra $\mathcal{U}(A)$.

Recall that the universal enveloping algebra $\mathcal{U}(A)$ of a Lie algebroid A over M is simultaneously an associative algebra and an R -bimodule. In case the Lie algebroid A is real, $\mathcal{U}(A)$ is canonically identified to the algebra of left-invariant s-fiberwise differential operators on the local Lie groupoid \mathcal{A} integrating A . Let us recall its construction.

The vector space $\mathfrak{g} = R \oplus \Gamma(A)$ admits a natural Lie algebra structure given by the Lie bracket

$$[f + X, g + Y] = \rho(X)g - \rho(Y)f + [X, Y],$$

where $f, g \in R$ and $X, Y \in \Gamma(A)$. Here ρ denotes the anchor map. Let i denote the natural inclusion of \mathfrak{g} into its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The universal enveloping algebra $\mathcal{U}(A)$ of the Lie algebroid A is the quotient of the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $i(\mathfrak{g})$ by the two-sided ideal generated by the elements of the form $i(f) \otimes i(g + Y) - i(fg + fY)$ with $f, g \in R$ and $Y \in \Gamma(A)$.

When A is a Lie algebra, $\mathcal{U}(A)$ is indeed the usual universal enveloping algebra. On the other hand, when A is the tangent bundle TM , $\mathcal{U}(A)$ is the algebra of differential operators on M .

We use the symbol \mathcal{A} to denote the abelian category of A -modules. Abusing terminology, we say that a vector bundle E over M is an A -module if $\Gamma(E) \in \mathcal{A}$.

Given a Lie pair (L, A) of algebroids, i.e. a Lie algebroid L with a Lie subalgebroid A , the Atiyah class α_E of an A -module E relative to the pair (L, A) is defined as the obstruction to the existence of an A -compatible L -connection on the vector bundle E . An L -connection ∇ on an A -module E is said to be A -compatible if it extends the given flat A -connection on E and satisfies $\nabla_a \nabla_l - \nabla_l \nabla_a = \nabla_{[a, l]}$ for all $a \in \Gamma(A)$ and $l \in \Gamma(L)$. This fairly recently defined class (see [1]) has as double

Research partially supported by NSF grant DMS1101827, NSA grant H98230-12-1-0234, and NSFC grants 11001146 and 11471179.

origin, which it generalizes, the Atiyah class of holomorphic vector bundles and the Molino class of foliations.

The quotient L/A of any Lie pair (L, A) is an A -module [1]. Its Atiyah class $\alpha_{L/A}$ can be described as follows. Choose an L -connection ∇ on L/A extending the A -action. Its curvature is the vector bundle map $R^\nabla : \wedge^2 L \rightarrow \text{End}(E)$ defined by $R^\nabla(l_1, l_2) = \nabla_{l_1} \nabla_{l_2} - \nabla_{l_2} \nabla_{l_1} - \nabla_{[l_1, l_2]}$, for all $l_1, l_2 \in \Gamma(L)$. Since L/A is an A -module, R^∇ vanishes on $\wedge^2 A$ and, therefore, determines a section $R_{L/A}^\nabla$ of $A^* \otimes (L/A)^* \otimes \text{End}(L/A)$. It was proved in [1] that $R_{L/A}^\nabla$ is a 1-cocycle for the Lie algebroid A with values in the A -module $(L/A)^* \otimes \text{End}(L/A)$ and that its cohomology class $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \text{End}(L/A))$ is independent of the choice of the connection.

Let $\text{Ch}^b(\mathcal{A})$ denote the category of bounded complexes in \mathcal{A} and let $D^b(\mathcal{A})$ denote the corresponding derived category. We write $L/A[-1]$ to denote the quotient L/A regarded as a complex in \mathcal{A} concentrated in degree 1.

The following was proved in [1].

Proposition 1.1 ([1]). *Let (L, A) be a Lie algebroid pair. The Atiyah class $\alpha_{L/A}$ of the quotient L/A relative to the pair (L, A) determines a morphism*

$$L/A[-1] \otimes L/A[-1] \rightarrow L/A[-1]$$

in the derived category $D^b(\mathcal{A})$ making $L/A[-1]$ a Lie algebra object in $D^b(\mathcal{A})$.

It is well known that every ordinary Lie algebra \mathfrak{g} admits a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is a Hopf algebra. We are thus led to the following natural questions: does there exist a universal enveloping algebra for $L/A[-1]$ in $D^b(\mathcal{A})$ and, if so, is it a Hopf algebra object?

In this Note, we give a positive answer to the questions above. For a complex manifold X , the Atiyah class of the Lie pair $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is simply the usual Atiyah class of the holomorphic tangent bundle T_X recently exploited by Kapranov [2]. It was proved that the universal enveloping algebra of the Lie algebra object $T_X[-1]$ in $D^b(X)$ is the Hochschild cochain complex $(\mathcal{D}_{\text{poly}}^\bullet(X), d)$ [5, 6, 7]. This result played an important role in the study of several aspects of complex geometry including the Riemann-Roch theorem [5], the Chern character [6] and the Rozansky-Witten invariants [7, 8]. Applications of our result will be developed elsewhere.

2. HOCHSCHILD-KOSTANT-ROSENBERG MAP

It is known [9] that the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebroid L admits a cocommutative coassociative coproduct $\Delta : \mathcal{U}(L) \rightarrow \mathcal{U}(L) \tilde{\otimes} \mathcal{U}(L)$, which is defined on generators as follows: $\Delta(f) = f \tilde{\otimes} 1 = 1 \tilde{\otimes} f, \forall f \in R$ and $\Delta(l) = l \tilde{\otimes} 1 + 1 \tilde{\otimes} l, \forall l \in \Gamma(L)$. Here, and in the sequel, $\tilde{\otimes}$ stands for the tensor product of left R -modules. Moreover, $\mathcal{U}(L)$ is an L -module since each section l of L acts on $\mathcal{U}(L)$ by left multiplication: $\nabla_l u = l \cdot u, \forall u \in \mathcal{U}(L)$.

Now, given a Lie pair (L, A) , consider the quotient $\mathcal{D}_{\text{poly}}^1$ of $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. It is straightforward to see that the comultiplication on $\mathcal{U}(L)$ induces a comultiplication $\Delta : \mathcal{D}_{\text{poly}}^1 \rightarrow \mathcal{D}_{\text{poly}}^1 \tilde{\otimes} \mathcal{D}_{\text{poly}}^1$ on $\mathcal{D}_{\text{poly}}^1$ and the action of L on $\mathcal{U}(L)$ determines an action of A on $\mathcal{D}_{\text{poly}}^1$.

Lemma 2.1. *The quotient $\mathcal{D}_{\text{poly}}^1 = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is simultaneously a cocommutative coassociative R -coalgebra and an A -module. Moreover, its comultiplication is compatible with its A -action:*

$$\nabla_X(\Delta p) = \Delta(\nabla_X p), \quad \forall X \in \Gamma(A), p \in \mathcal{D}_{\text{poly}}^1.$$

Let $\mathcal{D}_{\text{poly}}^n$ denote the n -th tensorial power $\mathcal{D}_{\text{poly}}^1 \tilde{\otimes} \cdots \tilde{\otimes} \mathcal{D}_{\text{poly}}^1$ of $\mathcal{D}_{\text{poly}}^1$ and, for $n = 0$, set $\mathcal{D}_{\text{poly}}^0 = R$. We define a coboundary operator $d : \mathcal{D}_{\text{poly}}^\bullet \rightarrow \mathcal{D}_{\text{poly}}^{\bullet+1}$ on $\mathcal{D}_{\text{poly}}^\bullet = \bigoplus_{n=0}^\infty \mathcal{D}_{\text{poly}}^n$ by

$$\begin{aligned} d(p_1 \tilde{\otimes} \cdots \tilde{\otimes} p_n) &= 1 \tilde{\otimes} p_1 \tilde{\otimes} \cdots \tilde{\otimes} p_n - (\Delta p_1) \tilde{\otimes} \cdots \tilde{\otimes} p_n + p_1 \tilde{\otimes} (\Delta p_2) \tilde{\otimes} \cdots \tilde{\otimes} p_n - \cdots \\ &\quad + (-1)^n p_1 \tilde{\otimes} \cdots \tilde{\otimes} p_{n-1} \tilde{\otimes} (\Delta p_n) + (-1)^{n+1} p_1 \tilde{\otimes} \cdots \tilde{\otimes} p_n \tilde{\otimes} 1, \end{aligned} \quad (1)$$

for any $p_1, p_2, \dots, p_n \in \mathcal{D}_{\text{poly}}^1$. Since the comultiplication Δ is compatible with the action of A , the operator d is a morphism of A -modules. Moreover, Δ being coassociative, d satisfies $d^2 = 0$. Thus $(\mathcal{D}_{\text{poly}}^\bullet, d)$ is an object of $\text{Ch}^b(\mathcal{A})$.

When endowed with the trivial coboundary operator, the space of sections of

$$S^\bullet(L/A[-1]) = \bigoplus_{k=0}^\infty S^k(L/A[-1]) = \bigoplus_{k=0}^\infty (\wedge^k L/A)[-k]$$

is a complex of A -modules:

$$0 \rightarrow R \xrightarrow{0} \Gamma(L/A) \xrightarrow{0} \Gamma(\wedge^2(L/A)) \xrightarrow{0} \Gamma(\wedge^3(L/A)) \xrightarrow{0} \cdots$$

The natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}_{\text{poly}}^1$ extends naturally to the Hochschild-Kostant-Rosenberg map

$$\text{HKR} : \Gamma(S^\bullet(L/A[-1])) \rightarrow \mathcal{D}_{\text{poly}}^\bullet$$

by skew-symmetrization:

$$\begin{aligned} \text{HKR}(b_1 \wedge \cdots \wedge b_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{\sigma(1)} \tilde{\otimes} b_{\sigma(2)} \tilde{\otimes} \cdots \tilde{\otimes} b_{\sigma(n)}, \\ &\quad \forall b_1, \dots, b_n \in \Gamma(L/A). \end{aligned} \quad (2)$$

Proposition 2.2. *In $\text{Ch}^b(\mathcal{A})$, the Hochschild-Kostant-Rosenberg map is a quasi-isomorphism from $(\Gamma(S^\bullet(L/A[-1])), 0)$ to $(\mathcal{D}_{\text{poly}}^\bullet, d)$.*

Sketch of proof. Assuming L and A are real Lie algebroids, let \mathcal{L} and \mathcal{A} be local Lie groupoids integrating L and A respectively. The source map $s : \mathcal{L} \rightarrow M$ induces a surjective submersion $J : \mathcal{L}/\mathcal{A} \rightarrow M$. The right quotient \mathcal{L}/\mathcal{A} is a left \mathcal{L} -homogeneous space with momentum map J [4]. Therefore, it admits an infinitesimal L -action, and hence an infinitesimal A -action. The coalgebra $\mathcal{D}_{\text{poly}}^1$ may be regarded as the space of distributions on the J -fibers of \mathcal{L}/\mathcal{A} supported on M . Its A -module structure then stems from the infinitesimal A -action on \mathcal{L}/\mathcal{A} . The n -th tensorial power $\mathcal{D}_{\text{poly}}^n$ may be viewed as the space of n -differential operators on the J -fibers of \mathcal{L}/\mathcal{A} evaluated along M and the differential d as the Hochschild coboundary. The conclusion follows from the classical Hochschild-Kostant-Rosenberg theorem. To prove the proposition for complex Lie algebroids, it suffices to consider formal groupoids instead of local Lie groupoids [3].

3. UNIVERSAL ENVELOPING ALGEBRA OF $L/A[-1]$ IN $D^b(\mathcal{A})$

Following Markarian [5], Ramadoss [6], and Roberts-Willerton [7], we introduce the following:

Definition 3.1. *If it exists, the universal enveloping algebra of a Lie algebra object \mathcal{G} in $D^b(\mathcal{A})$ is an associative algebra object \mathcal{H} in $D^b(\mathcal{A})$ together with a morphism of Lie algebras $i : \mathcal{G} \rightarrow \mathcal{H}$ satisfying the following universal property: given any associative algebra object \mathcal{K} and any morphism of Lie algebras $f : \mathcal{G} \rightarrow \mathcal{K}$ in $D^b(\mathcal{A})$, there exists a unique morphism of associative algebras $f' : \mathcal{H} \rightarrow \mathcal{K}$ in $D^b(\mathcal{A})$ such that $f = f' \circ i$.*

In view of the similarity between $(\mathcal{D}_{\text{poly}}^\bullet, d)$ and the Hochschild cochain complex, we define a cup product \cup on $\mathcal{D}_{\text{poly}}^\bullet$ by setting $P \cup Q = P \tilde{\otimes} Q$, for all $P, Q \in \mathcal{D}_{\text{poly}}^\bullet$. It is simple to check that

$$d(P \cup Q) = dP \cup Q + (-1)^{|P|} P \cup dQ,$$

for all homogeneous $P, Q \in \mathcal{D}_{\text{poly}}^\bullet$.

Proposition 3.2. *For any Lie pair (L, A) of algebroids, $(\mathcal{D}_{\text{poly}}^\bullet, d, \cup)$ is an associative algebra object in $D^b(\mathcal{A})$, which is in fact the universal enveloping algebra of the Lie algebra $L/A[-1]$ in $D^b(\mathcal{A})$.*

Consider the inclusion $\eta : R \hookrightarrow \mathcal{D}_{\text{poly}}^n$, the projection $\varepsilon : \mathcal{D}_{\text{poly}}^n \twoheadrightarrow R$, and the maps $t : \mathcal{D}_{\text{poly}}^\bullet \rightarrow \mathcal{D}_{\text{poly}}^\bullet$ and $\tilde{\Delta} : \mathcal{D}_{\text{poly}}^\bullet \rightarrow \mathcal{D}_{\text{poly}}^\bullet \otimes_R \mathcal{D}_{\text{poly}}^\bullet$ defined, respectively, by

$$t(p_1 \tilde{\otimes} p_2 \tilde{\otimes} \cdots \tilde{\otimes} p_n) = (-1)^{\frac{n(n-1)}{2}} p_n \tilde{\otimes} p_{n-1} \tilde{\otimes} \cdots \tilde{\otimes} p_1$$

and

$$\tilde{\Delta}(p_1 \tilde{\otimes} p_2 \tilde{\otimes} \cdots \tilde{\otimes} p_n) = \sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}_i^j} \text{sgn}(\sigma) (p_{\sigma(1)} \tilde{\otimes} \cdots \tilde{\otimes} p_{\sigma(i)}) \otimes (p_{\sigma(i+1)} \tilde{\otimes} \cdots \tilde{\otimes} p_{\sigma(n)}),$$

where \mathfrak{S}_i^j denotes the set of (i, j) -shuffles.¹

Theorem 3.3. *For any Lie pair (L, A) of algebroids, $(\mathcal{D}_{\text{poly}}^\bullet, d)$ with the multiplication \cup , the comultiplication $\tilde{\Delta}$, the unit η , the counit ε , and the antipode t , is a Hopf algebra object in $D^b(\mathcal{A})$.*

4. RAMADOSS'S APPROACH: $L(\mathcal{D}_{\text{poly}}^1)$

To prove Proposition 3.2 and Theorem 3.3, we essentially follow Ramadoss's approach [6]. Let $L(\mathcal{D}_{\text{poly}}^1)$ be the (graded) free Lie algebra generated over R by $\mathcal{D}_{\text{poly}}^1$ concentrated in degree 1. In other words, $L(\mathcal{D}_{\text{poly}}^1)$ is the smallest Lie subalgebra of $\mathcal{D}_{\text{poly}}^\bullet$ containing $\mathcal{D}_{\text{poly}}^1$. The Lie bracket of two vectors $u \in \mathcal{D}_{\text{poly}}^i$ and $v \in \mathcal{D}_{\text{poly}}^j$ is the vector $[u, v] = u \tilde{\otimes} v - (-1)^{ij} v \tilde{\otimes} u \in \mathcal{D}_{\text{poly}}^{i+j}$. Actually, $L(\mathcal{D}_{\text{poly}}^1)$ is made of all linear combinations of elements of the form $[p_1, [p_2, [\cdots, [p_{n-1}, p_n] \cdots]]$ with $p_1, \dots, p_n \in \mathcal{D}_{\text{poly}}^1$. One checks that $L(\mathcal{D}_{\text{poly}}^1)$ is a d -stable A -submodule of $\mathcal{D}_{\text{poly}}^\bullet$ and that its Lie bracket is a chain map with respect to the coboundary operator d . Therefore $(L(\mathcal{D}_{\text{poly}}^1), d)$ is a Lie algebra object in $\text{Ch}^b(\mathcal{A})$.

¹An (i, j) -shuffle is a permutation σ of the set $\{1, 2, \dots, i+j\}$ such that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(i)$ and $\sigma(i+1) \leq \sigma(i+2) \leq \cdots \leq \sigma(i+j)$.

Let $S^\bullet(L(\mathcal{D}_{\text{poly}}^1))$ be the symmetric algebra of $L(\mathcal{D}_{\text{poly}}^1)$ and let

$$I : S^\bullet(L(\mathcal{D}_{\text{poly}}^1)) \rightarrow \mathcal{D}_{\text{poly}}^\bullet$$

be the symmetrization map:

$$I(z_1 \odot \cdots \odot z_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma; z_1, \dots, z_n) z_{\sigma(1)} \tilde{\otimes} z_{\sigma(2)} \tilde{\otimes} \cdots \tilde{\otimes} z_{\sigma(n)}.$$

The Koszul sign $\text{sgn}(\sigma; z_1, \dots, z_n)$ of a permutation σ of the (homogeneous) vectors $z_1, z_2, \dots, z_n \in S^\bullet(L(\mathcal{D}_{\text{poly}}^1))$ is determined by the relation

$$z_{\sigma(1)} \odot z_{\sigma(2)} \odot \cdots \odot z_{\sigma(n)} = \text{sgn}(\sigma; z_1, \dots, z_n) z_1 \odot z_2 \odot \cdots \odot z_n.$$

Lemma 4.1. *The symmetrization $I : S^\bullet(L(\mathcal{D}_{\text{poly}}^1)) \rightarrow \mathcal{D}_{\text{poly}}^\bullet$ is an isomorphism in $\text{Ch}^b(\mathcal{A})$.*

Using Lemma 4.1 and the HKR quasi-isomorphism, one can prove that the composition $\beta : \Gamma(L/A[-1]) \rightarrow L(\mathcal{D}_{\text{poly}}^1)$ of the inclusions

$$\Gamma(L/A[-1]) \subset \mathcal{D}_{\text{poly}}^1 \subset L(\mathcal{D}_{\text{poly}}^1)$$

is a quasi-isomorphism in $\text{Ch}^b(\mathcal{A})$, which intertwines the Lie brackets on $\Gamma(L/A[-1])$ and $L(\mathcal{D}_{\text{poly}}^1)$.

Proposition 4.2. (1) *The inclusion $\beta : \Gamma(L/A[-1]) \rightarrow L(\mathcal{D}_{\text{poly}}^1)$ is a quasi-isomorphism in $\text{Ch}^b(\mathcal{A})$.*
 (2) *The inclusion $\beta : \Gamma(L/A[-1]) \rightarrow L(\mathcal{D}_{\text{poly}}^1)$ is an isomorphism of Lie algebra objects in $D^b(\mathcal{A})$ as the diagram*

$$\begin{array}{ccc} \Gamma(L/A[-1]) \tilde{\otimes} \Gamma(L/A[-1]) & \xrightarrow{\beta \otimes \beta} & L(\mathcal{D}_{\text{poly}}^1) \tilde{\otimes} L(\mathcal{D}_{\text{poly}}^1) \\ \alpha_{L/A} \downarrow & & \downarrow [\cdot, \cdot] \\ \Gamma(L/A[-1]) & \xrightarrow{\beta} & L(\mathcal{D}_{\text{poly}}^1) \end{array}$$

commutes in $D^b(\mathcal{A})$.

Proposition 3.2 and Theorem 3.3 now follow immediately.

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DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, CHINA
E-mail address: `zchen@math.tsinghua.edu.cn`

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNITED STATES
E-mail address: `stienon@psu.edu`

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNITED STATES
E-mail address: `ping@math.psu.edu`